Filomat 29:4 (2015), 829–838
DOI 10.2298/FIL1504829G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Advances on Ricceri's Most Famous Conjecture

F. J. García-Pacheco[a], J. R. Hill[b]

[a] *Department of Mathematical Sciences, University of Cadiz, Puerto Real, Spain, 11510, EU*
[b] *Department of Mathematics and Physics, Texas A&M University-Central Texas, Killeen, TX, 76549, USA*

Abstract. New advances towards a (positive) solution to Ricceri's (most famous) Conjecture are presented. One of these advances consists of showing that a totally anti-proximinal absolutely convex subset of a vector space is linearly open. We also prove that if every totally anti-proximinal convex subset of a vector space is linearly open then Ricceri's Conjecture holds true. Finally we demonstrate that the concept of total anti-proximinality does not make sense in the scope of pseudo-normed spaces.

1. Introduction

In [8] Ricceri establishes the notion of total anti-proximinality and poses a conjecture on the topological structure of such sets.

Definition 1.1 (Ricceri, [8]).

- *Let E be a metric space. A non-empty proper subset A of E is called anti-proximinal exactly when for every element $e \in E \setminus A$ the distance from e to A, $d(e, A)$, is never attained at any $a \in A$.*
- *Let E be a vector space. A non-empty proper subset A of E is called totally anti-proximinal exactly when A is anti-proximinal for every norm on E.*
- *A Hausdorff locally convex topological vector space E is said to have the anti-proximinal property if every totally anti-proximinal convex subset is not rare.*
- *A Hausdorff locally convex topological vector space E is said to have the quasi anti-proximinal property if every totally anti-proximinal quasi-absolutely convex subset is not rare.*
- *A Hausdorff locally convex topological vector space E is said to have the weak anti-proximinal property if every totally anti-proximinal absolutely convex subset is not rare.*

In order to form a better understanding of the previous definition, we remind the reader about the following concepts:

- A subset of a topological space is said to be rare exactly when its closure has empty interior.

2010 *Mathematics Subject Classification*. Primary 15A03; Secondary 46A55, 46B20
Keywords. anti-proximinal, barrelled, Ricceri, Baire, absorbing, linearly open
Received: 24 October 2013; Accepted: 03 December 2013
Communicated by Mohammad Sal Moslehian
Email addresses: garcia.pacheco@uca.es (F. J. García-Pacheco), justin.hill@ct.tamus.edu (J. R. Hill)

- A quasi-absolutely convex set is a convex set containing 0 which also contains a translate of a negative multiple of itself (see Definition 3.1).
- Absolute convexity implies quasi-absolute convexity which implies convexity.
- The anti-proximinal property implies the quasi anti-proximinal property which implies the weak anti-proximinal property.

1.1. *Ricceri's (most famous) Conjecture*

With these definitions in mind, Ricceri proposed the following conjecture.

Conjecture 1.2 (Ricceri's Conjecture, [8]). *There exists a non-complete normed space enjoying the anti-proximinal property.*

The following theorem compiles a series of results from [5, 7] about Ricceri's Conjecture.

Theorem 1.3 (Garcia-Pacheco and Hill, [5, 7]). *Let X be a Hausdorff locally convex topological vector space. The following conditions are equivalent:*

- *X satisfies the weak anti-proximinal property.*
- *X satisfies the quasi anti-proximinal property.*
- *X is barrelled.*

As well, in [5] the following partial positive solution to Ricceri's Conjecture is provided.

Corollary 1.4 (Garcia-Pacheco, [5]). *There exists a non-complete normed space enjoying the weak anti-proximinal property.*

1.2. *Notation*

To end this introduction, we introduce the proper notation used throughout this manuscript.

- Given a vector space X, the linear span or vector subspace generated by a subset A of X will be denoted by $\operatorname{span}(A)$.
- Given a vector space X and non-empty subset A of X, $\operatorname{bl}(A), \operatorname{co}(A)$, and $\operatorname{aco}(A)$ will denote the balanced hull, the convex hull, and the absolutely convex hull of A, respectively.
- If X is now a topological space and A is a subset of X, then $\operatorname{int}(A)$, $\operatorname{cl}(A)$, and $\operatorname{bd}(A)$ will denote the topological interior, the topological closure, and the topological boundary of A, respectively.
- Given a normed space X, the open unit ball of X will be denoted by U_X, the closed unit ball or simply the unit ball of X will be denoted by B_X, and the unit sphere of X is S_X.

2. A Simpler Reformulation of [5, Theorem 3.6]

In this section we will reprove Theorem 3.6 from [5] with weaker hypotheses, as it will be useful.

Definition 2.1. *Let X be a vector space. A non-empty subset A of X is said to almost contain 0 if there exists $a_0 \in A$ with $a_0 - A \subseteq \operatorname{bl}(A)$.*

It is fairly obvious that if a set contains 0, then it almost contains 0. The converse does not trivially hold since it only suffices to consider in $\mathbb{R}$ any interval not containing 0.

Theorem 2.2. *Let X be a normed space. Let A be a non-empty subset of X that almost contains* 0. *If A is anti-proximinal, then* $\mathrm{bl}(A)$ *and* $\mathrm{aco}(A)$ *are also anti-proximinal.*

Proof. First off, observe that in virtue of [5, Theorem 3.6] it only suffices to prove that $\mathrm{bl}(A)$ is anti-proximinal. Let $x \notin \mathrm{bl}(A)$ and suppose that there exists $\gamma \in \mathsf{B}_{\mathbb{K}}$ and $a \in A$ such that $d(x, \mathrm{bl}(A)) = \left\|x - \gamma a\right\|$. We will distinguish between two cases:

- $\gamma = 0$. In this case we have that $\|x\| = d(x, \mathrm{bl}(A)) \leq \|x - \lambda b\|$ for all $b \in A$ and all $\lambda \in \mathsf{B}_{\mathbb{K}}$. By hypothesis we can find $a_0 \in A$ with $a_0 - A \subseteq \mathrm{bl}(A)$. We will show that $y \notin A$ and $\left\|y - a_0\right\| = d(y, A)$, where $y := x + a_0$. Assume first that there exists $b \in A$ such that $x + a_0 = b$. Then $x = -(a_0 - b) \in -\mathrm{bl}(A) = \mathrm{bl}(A)$, which is not possible. Now consider any $b \in A$. By hypothesis, $a_0 - b \in \mathrm{bl}(A)$, therefore $\left\|y - b\right\| = \|x + (a_0 - b)\| \geq \|x\| = \left\|y - a_0\right\|$. This shows that A is not anti-proximinal.

- $\gamma \neq 0$. Observe that $\frac{1}{\gamma}x \notin A$ and

  $$d\left(\frac{1}{\gamma}x, A\right) = \left\|\frac{1}{\gamma}x - a\right\|,$$

  which means that A is not anti-proximinal.

□

The reader may notice that the hypothesis of A almost containing 0 has only been used in the first part of the proof of the previous theorem.

Corollary 2.3. *Let X be a vector space. Let A be a non-empty subset of X that almost contains* 0. *If A is a totally anti-proximinal subset of X, then* $\mathrm{bl}(A)$ *and* $\mathrm{aco}(A)$ *are also totally anti-proximinal.*

3. Using Inner Structure to Construct a Convex Set Containing 0 Which is not Quasi-Absolutely Convex

The reader may notice that one possible way to prove Ricceri's Conjecture right is by showing that every totally anti-proximinal convex set containing 0 is quasi-absolutely convex (because in this situation the anti-proximinal property would be equivalent to the quasi anti-proximinal property, which is itself equivalent to the weak anti-proximinal property in virtue of Theorem 1.3, and then we would simply need to apply Corollary 1.4). The purpose of this section is to construct an example of a convex set containing 0 which is not quasi-absolutely convex. We will first recall the concept of quasi-absolute convexity (see [7, Definition 2.1]).

Definition 3.1 (Garcia-Pacheco and Hill, [7]). *Let X be a vector space. Let A be a non-empty subset of X.*

- *We will say that A is quasi-balanced if there exist $a \in A$ and $\varepsilon\,(0, 1)$ such that $a - \varepsilon A \subseteq A$.*
- *We will say that A is quasi-absolutely convex if it is convex, quasi-balanced, and* $0 \in A$.

Remark 3.2. *Let X be a vector space and consider a non-empty subset A of X:*

1. *If A is convex and* $0 \in A$, *then* $[0, 1]\,A \subseteq A$ *and* $A + A = 2A$.
2. *If A is absolutely convex, then A is quasi-absolutely convex. Indeed, A is convex and* $[-1, 1]\,A \subseteq A$, *therefore* $0 - \varepsilon A \subseteq A$ *for all* $\varepsilon \in (0, 1)$.

The reader may find it easy to construct quasi-absolutely convex sets which are not absolutely convex. As we have already mentioned, this section is devoted to show the existence of a convex set containing 0 which is not quasi-absolutely convex.

3.1. Inner structure

We will need the concept of inner point, which was first introduced in [3].

Definition 3.3 (Garcia-Pacheco, [3]). *Let X be a real vector space and consider M a non-empty subset of X. Let $x \in X$.*

1. *We will say that x is an inner point of M provided that the following happens: Let S be a bounded or unbounded maximal segment of M such that $x \in S$, then $x \in \operatorname{int}(S)$.*
2. *We will say that x is an outer point of M provided that the following happens: There exists a bounded or unbounded maximal segment S of M such that $x \in \operatorname{ext}(S)$.*

The set of inner and outer points of M will be denoted by $\operatorname{inn}(M)$ and $\operatorname{out}(M)$, respectively. The reader may observe that $\{\operatorname{inn}(M), \operatorname{out}(M) \cap M\}$ is always a partition of M. Even more, if M is convex, then

- $\operatorname{out}(M) = \operatorname{out}(X \setminus M)$, and
- $\{\operatorname{inn}(M), \operatorname{out}(M), \operatorname{inn}(X \setminus M)\}$ is a partition of X.

In case M is not convex, then the previous two items do not hold as shown in the next example.

Example 3.4. *Let $X := \mathbb{R}^2$ and consider*

$$M := (-\infty, 0) \times \{0\} \cup \left\{ \left(\frac{1}{n}, 0\right) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}.$$

It is pretty clear that $0 \in \operatorname{out}(M) \setminus M$. However, the only maximal segments of $X \setminus M$ that contain 0 are the straight lines $\{(x, mx) : x \in \mathbb{R}\}$ with $m \neq 0$. As a consequence, $0 \in \operatorname{inn}(X \setminus M)$.

In case X is endowed with a vector topology, then the relation between the inner and outer points and the topological interior and closure is the following for convex subsets (see [3, Chapter 1]):

- $\operatorname{int}(M) \subseteq \operatorname{inn}(M)$.
- If $\operatorname{int}(M) \neq \varnothing$, then $\operatorname{int}(M) = \operatorname{inn}(M)$.
- $\operatorname{out}(M) \subseteq \operatorname{bd}(M)$.

3.2. An example of a convex set containing 0 which is not quasi-absolutely convex

The main theorem in this section follows.

Theorem 3.5. *Let X be a vector space. If A is a quasi-absolutely convex subset of X, then $\operatorname{inn}(A) \neq \varnothing$.*

Proof. Let $a \in A$ and $0 < \varepsilon < 1$ such that $a - \varepsilon A \subseteq A$. We will show that $\frac{a}{2} \in \operatorname{inn}(A)$. Let $b \in A$. There exists $c \in A$ such that $a - \varepsilon b = c$. Then

$$\frac{a}{2} = \frac{\varepsilon}{2} b + \left(1 - \frac{\varepsilon}{2}\right) \frac{c}{\left(1 - \frac{\varepsilon}{2}\right) 2}.$$

Finally notice that

$$\frac{c}{\left(1 - \frac{\varepsilon}{2}\right) 2} \in A$$

by bearing in mind Remark 3.2 and the fact that $\left(1 - \frac{\varepsilon}{2}\right) 2 > 1$. □

Example 3.6 (A convex set containing 0 which is not quasi-absolutely convex). *In [3, Corollary 1.2.10] and in [6, Theorem 3.2] examples of non-trivial convex sets free of inner points are given. Any translate of them containing 0 is still free of inner points and constitutes and example of a convex set containing 0 which is not quasi-absolutely convex in virtue of Theorem 3.5.*

4. A New Approach to a Positive Solution to Ricceri's Conjecture

In this section we will schematize a new and different approach towards achieving the truthfulness of Ricceri's Conjecture. Basically, everything reduces to showing the following crucial fact: *every totally anti-proximinal convex set has an absorbing translate.* Proving this fact right conveys to proving Ricceri's Conjecture true. Everything begins with the following lemma.

Lemma 4.1. *Let X be a Hausdorff locally convex topological vector space. Let A be an absorbing subset of X. If X is a Baire space, then A is not rare.*

Proof. Since A is absorbing we have that $X = \bigcup_{n\in\mathbb{N}} nA$. By hypothesis, there exists $n \in \mathbb{N}$ so that nA is non-rare, and so is A. □

4.1. Proving Ricceri's Conjecture true under the assumption that every totally anti-proximinal convex set has an absorbing translate

In the whole of this subsection we will assume that *every totally anti-proximinal convex set has an absorbing translate.*

Theorem 4.2. *Let X be a Hausdorff locally convex topological vector space. If X is a Baire space, then X satisfies the anti-proximinal property.*

Proof. Let A be a totally anti-proximinal convex subset of X. By our assumption we deduce the existence of a translate of A which is absorbing. In virtue of Lemma 4.1 we have that that translate of A is non-rare, and so is A. □

Theorem 4.2 together with several classic results (and our assumption) will give us the key to prove Ricceri's Conjecture true.

Example 4.3 (Positive Solution to Ricceri's Conjecture). There exists a non-complete normed space satisfying the anti-proximinal property. *Indeed, it suffices to consider any non-complete normed space which is a Baire space and apply Theorem 4.2. For an example of a non-complete normed space which is Baire take a look at [2, Chapter 3] where it is observed that if E is a separable, infinite-dimensional Banach space, then E contains a dense subspace M of countably infinite codimension which is a Baire space.*

4.2. Possible pathway for proving the assumption that every totally anti-proximinal convex set has an absorbing translate

In this subsection we trace a path which could lead to proving that *every totally anti-proximinal convex set has an absorbing translate*. Notice that it suffices to show that every totally anti-proximinal convex set is linearly open, that is, composed only of internal points, or equivalently, open in the finest locally convex vector topology (see [2]). At this stage, it is necessary to remind the reader about the concept of "internal point" (see [2]). Given a vector space E and a non-empty subset A of E, we say that $a \in E$ is an internal point of A when every straight line passing through a has a small interval around a entirely contained in A. More precisely, $a \in E$ is an internal point of A when, for every $x \in E$, there exists $\delta_x > 0$ such that $a + \lambda x \in A$ for all $\lambda \in [0, \delta_x]$. The set of internal points of A is denoted by $\operatorname{inter}(A)$. Well known facts about internal points, which can be consulted in [2], are the following:

- In any topological vector space if a subset A is (absolutely) convex and $\operatorname{inter}(A) \neq \varnothing$, then $\operatorname{inter}(A)$ is also (absolutely) convex and $\operatorname{cl}(\operatorname{inter}(A)) = \operatorname{cl}(A)$.
- Every open set of any topological vector space is composed of internal points.
- The finest locally convex vector topology on a given vector space is the one whose basis of open sets are exactly the convex sets composed only of internal points.

In relation to this we refer the reader to [5, Theorem 4.1], which follows:

Theorem 4.4 (García-Pacheco, [5]). *Let E be a vector space. Let A be a non-empty subset of E.*

1. *If $A = \text{inter}(A)$, then A is totally anti-proximinal.*
2. *Conversely, if A is totally anti-proximinal, absolutely convex and contains no half-line of E, then $A = \text{inter}(A)$.*

So the appropriate steps to follow in order to show that every totally anti-proximinal convex set is linearly open (and thus it has an absorbing translate) are the following:

1. Removing the hypothesis of linear boundedness from (2) of Theorem 4.4.
2. Removing the hypothesis of balancedness from (2) of Theorem 4.4.

The first step is fully accomplished in this manuscript.

5. Removing the Hypothesis of Linear Boundedness from (2) of Theorem 4.4

The hypothesis of linear boundedness is strongly linked to working with semi-norms which are not norms. This is why we will divide this section into three subsections:

- In the first one we will characterize the linear boundedness in the class of absolutely convex sets.
- In the second one we will gather all the necessary tools to accomplish our objective in this section and, as a consequence of this gathering, we will also prove that the concept of total anti-proximinality does not make sense for pseudo-normed spaces.
- In the third one we will prove a generalized version of (2) in Theorem 4.4 with the hypothesis of linear boundedness removed.

5.1. *A characterization of linear boundedness in the class of absolutely convex sets*

In the first place, we remind the reader that a subset of a vector space is said to be linearly bounded if it does not contain half-lines, or equivalent, every maximal segment of it is bounded. The reader may quickly notice that if a set is linearly bounded, then it does not contain non-trivial vector subspaces of X. The converse to this last assertion does not hold even under the hypothesis of balancedness. Indeed, it suffices to consider the set

$$\left\{(x,y) \in \mathbb{R}^2 : x < 0, y \in (-1,0)\right\} \cup \left\{(x,y) \in \mathbb{R}^2 : x > 0, y \in (0,1)\right\} \cup \{(0,0)\},$$

which is balanced and does not contain non-trivial vector subspaces of $\mathbb{R}^2$ but it is not linearly bounded. What we will show next is that the converse to the assertion mentioned above does hold if the set is absolutely convex.

Proposition 5.1. *Let X be a vector space. If A is an absolutely convex subset of X, then A is linearly bounded if and only if A contains no non-trivial vector subspaces of X.*

Proof. Assume that A is absolutely convex and contains no non-trvial vector subspaces of X. Suppose to the contrary that A is not linearly bounded and consider $a \neq b \in A$ such that $\{a + t(b-a) : t \in [0, \infty)\} \subseteq A$. It is not difficult to see that $\mathbb{K}(b-a) \subseteq A$. □

5.2. *Why total anti-proximinality is not defined for semi-normed spaces*

We will now focus on explaining why the concept of total anti-proximinality makes no sense in the scope of pseudo-normed spaces

Remark 5.2. *Let X be a semi-normed space. According to [1, Theorem 2.1] we have the following:*

- *The set $V := \{x \in X : \|x\| = 0\}$ is a closed vector subspace of X.*
- *For every $v \in V$ and every $x \in X$ we have that $\|v + x\| = \|x\|$.*

- *If $A \subseteq x + V$ for some $x \in X$, then $d(y, A) = \|y - x\|$ for all $y \in X$.*
- *If $A \subseteq \bigcup_{i \in I}(x_i + V)$ for some family $\{x_i : i \in I\} \subseteq X$, then $d(y, A) = \inf_{i \in I}\|y - x_i\|$ for all $y \in X$, and thus $d(y, A)$ is always attained provided that $\{x_i : i \in I\}$ is compact.*

Lemma 5.3. *Let X be a semi-normed space and consider $V = \{x \in X : \|x\| = 0\}$. Let A be a non-empty proper subset of X. If there exists $x \in X \setminus A$ such that $(x - A) \cap V \neq \varnothing$, then A is not anti-proximinal for the pseudo-metric given by the semi-norm.*

Proof. Simply notice that

$$0 \leq d(x, A) \leq \|x - a\| = 0,$$

where $a \in A$ is so that $x - a \in V$. □

Remark 5.4. *Let X be a semi-normed space. If $x \in X \setminus \mathsf{B}_X$, then*

$$d(x, \mathsf{B}_X) = \|x\| - 1 = \left\|x - \frac{x}{\|x\|}\right\|.$$

Indeed,

$$d(x, \mathsf{B}_X) \leq \left\|x - \frac{x}{\|x\|}\right\| = \|x\| - 1,$$

and if $y \in \mathsf{B}_X$, then

$$\|x\| - 1 \leq \|x\| - \|y\| \leq \left|\|x\| - \|y\|\right| \leq \|x - y\|.$$

Lemma 5.5. *Let X be a vector space. Let A be a totally anti-proximinal subset of X. Let $\|\cdot\|$ be a semi-norm on X. If there exists $e \in X \setminus A$ such that $d_{\|\cdot\|}(e, A) > 0$, then $d_{\|\cdot\|}(e, A)$ is not attained.*

Proof. Assume the existence of $a \in A$ so that $d_{\|\cdot\|}(e, A) = \|e - a\|$. Consider any norm $|\cdot|$ on X such that $|e - a| \leq \|e - a\|$. Define a new norm on X given by $\lfloor \cdot \rfloor := \max\{\|\cdot\|, |\cdot|\}$. Notice that

$$\lfloor e - a \rfloor \geq d_{\lfloor \cdot \rfloor}(e, A) \geq d_{\|\cdot\|}(e, A) = \|e - a\| = \lfloor e - a \rfloor,$$

which contradicts the fact that A is totally anti-proximinal. □

Proposition 5.6. *Let X be a vector space. Let A be a totally anti-proximinal subset of X. Let $\|\cdot\|$ be a semi-norm on X which is not a norm and consider $V = \{x \in X : \|x\| = 0\}$. The following conditions are equivalent:*

1. *A is anti-proximinal for the pseudo-metric given by the semi-norm $\|\cdot\|$.*
2. *For every $x \in X \setminus A$ we have that $(x - A) \cap V = \varnothing$.*

Proof.

(1)⇒(2) Let $x \in X \setminus A$ such that $(x - A) \cap V \neq \varnothing$. In virtue of Lemma 5.3 we deduce that A is not anti-proximinal for the pseudo-metric given by the semi-norm $\|\cdot\|$.

(2)⇒(1) Let $x \in X \setminus A$. All we need to show is that $d_{\|\cdot\|}(x, A)$ is not attained. Suppose to the contrary then that $d_{\|\cdot\|}(x, A)$ is attained. Bearing in mind Lemma 5.5, we may assume that $d_{\|\cdot\|}(x, A) = 0$. Then there exists $a \in A$ such that $0 = d_{\|\cdot\|}(x, A) = \|x - a\|$. This means that $x - a \in V$ and hence $(x - A) \cap V \neq \varnothing$.

□

Finally, we present the main result of this subsection which asserts that there are no totally anti-proximinal sets in the context of semi-normed spaces.

Corollary 5.7. *Let X be a vector space. No non-empty proper subset of X is anti-proximinal for every semi-norm defined on X.*

Proof. Suppose the existence of a non-empty proper subset A of X which is anti-proximinal for every semi-norm defined on X. Since every norm is a semi-norm, in particular we have that A is totally anti-proximinal. We will construct a semi-norm on X and find an element $x \in X \setminus A$ such that $(x - A) \cap V \neq \varnothing$, which will constitute a contradiction in virtue of Proposition 5.6. By hypothesis there exist $x \in X \setminus A$ and $a \in A$. At this stage it only suffices to consider any semi-norm on X whose set V of null-norm vectors contains $\mathbb{K}(x - a)$. □

5.3. *Removing the linear boundedness*

We will strongly rely on the results proven in the previous subsection.

Lemma 5.8. *Let E be a vector space. If A is totally anti-proximinal subset of A contained in the closed unit ball of a semi-norm on X, then A is actually contained in the open unit ball of that semi-norm.*

Proof. Let $\|\cdot\|$ be any semi-norm on X whose closed unit ball $\mathsf{B}_{\|\cdot\|}$ contains A. Suppose to the contrary that there exists $a \in A \cap \mathsf{S}_{\|\cdot\|}$. By applying Remark 5.4 we have that

$$1 = \|2a - a\| \geq d(2a, A) \geq d(2a, \mathsf{B}_{\|\cdot\|}) = \|2a\| - 1 = 1,$$

which contradicts Lemma 5.5. □

We remind the reader that a subset of a topological vector space is said to be finitely open provided that its intersection with every finite dimensional subspace is open in the Euclidean topology. According to [7, Theorem 3.2] a set is finitely open if and only if it is linearly open.

Theorem 5.9. *Let X be a vector space. Suppose that A is a totally anti-proximinal convex subset of X. Then the absolutely convex hull of A coincides with the open unit ball of the semi-norm that it generates and hence it is finitely open.*

Proof. It is well known that in this case and since A is convex, the absolutely convex hull of A is given by $\mathrm{co}(A \cup -A)$. Denote by $\|\cdot\|$ the semi-norm generated by the absolutely convex hull of A. Since

$$A, \mathsf{U}_{\|\cdot\|} \subseteq \mathrm{co}(A \cup -A) \subseteq \mathsf{B}_{\|\cdot\|},$$

by applying Lemma 5.8 we deduce that $A \subseteq \mathsf{U}_{\|\cdot\|}$, which automatically implies in virtue of the triangular inequality that $\mathrm{co}(A \cup -A) = \mathsf{U}_{\|\cdot\|}$. □

Corollary 5.10. *Let X be a vector space and A a non-empty proper subset of X. If A is totally anti-proximinal and absolutely convex, then A is finitely open.*

Hence we've removed the hypothesis of linear boundedness from (2) of Theorem 4.4.

Though we have met the goal of this section, we can go a little bit farther. It will not get us the complete removal of absolute convexity from (2) of Theorem 4.4, but it will get us close.

Corollary 5.11. *Let X be a vector space. Let A be a totally anti-proximinal subset of X. If $f : X \to \mathbb{R}$ is non-zero and linear, then* $\sup f(A)$ *is never attained.*

Proof. Suppose to the contrary that $a \in A$ is so that $f(a) = \sup f(A)$. Take any $x \in X \setminus A$ such that $f(x) > f(a)$. Consider the semi-norm on X given by $\|\cdot\| := \left|f(\cdot)\right|$. Since $0 < \left|f(x - a)\right| \leq \left|f(x - b)\right|$ for all $b \in A$, we deduce that $d(x, A) = \|x - a\|$, which contradicts Lemma 5.5. □

Scholium 5.12. *Let X be a vector space. If A is a totally anti-proximinal convex subset of X such that* $\mathrm{inter}(A) \neq \varnothing$, *then* $A = \mathrm{inter}(A)$, *that is, A is linearly open.*

Proof. Suppose to the contrary the existence of an element $a \in A \setminus \operatorname{inter}(A)$. Assume X endowed with the finest locally convex vector topology. In this situation, $\operatorname{int}(A) = \operatorname{inter}(A) \neq \varnothing$. According to the Hahn-Banach Separation Theorem, there exists $f \in X^* \setminus \{0\}$ such that $f(a) \geq \sup f(\operatorname{int}(A)) = \sup f(A)$, which indeed implies that $f(a) = \sup f(A)$. This fact contradicts Corollary 5.11. □

At the very end of the next section, which is the last one in the manuscript, it is shown that the previous scholium does not hold true if we remove the hypothesis of convexity. This section is also devoted to explain why the hypothesis of absolute convexity cannot be removed from (2) of Theorem 4.4.

6. Examples of Totally Anti-Proximinal Sets Which are not Linearly Open

We wish to dedicate this section to construct an example of a totally anti-proximinal set which is neither linearly open nor convex. In [5, Example 3.3] it is shown that any proper dense subset of a normed space is anti-proximinal. However, it does not necessarily have to be totally anti-proximinal. The main idea behind this fact is that changing the norm may change the density properties of the subset. This is why we need to consider "more general" dense sets, that is, sets that will be dense independently of the norm considered.

Remark 6.1. *Let X be a vector space. We will denote by $\mathbb{L}$ to the following fields: $\mathbb{Q}$ if X is real and $\mathbb{Q} + i\mathbb{Q}$ if X is complex. It is not difficult to check that if B is a Hamel basis for X, then*

$$Y := \left\{ l_1 b_1 + \cdots + l_p b_p : l_j \in \mathbb{L}, b_j \in B, 1 \leq j \leq p, p \in \mathbb{N} \right\}$$

is dense in X no matter what the vector topology X is endowed with. Notice that Y is a $\mathbb{Q}$-vector space.

Proposition 6.2. *Let X be a vector space. Let A be a non-empty linearly open subset of X. Then $A \cap Y$ is totally anti-proximinal but neither linearly open nor convex, where Y is the set considered in Remark 6.1.*

Proof. Firstly, notice that the density of Y in X endowed with the finest locally convex vector topology implies that $A \cap Y \neq \varnothing$. In fact, $A \cap Y$ is dense in A. Let $x \in X \setminus (A \cap Y)$ and $a \in A \cap Y$. Since A is linearly open, there exists $q \in (0,1) \cap \mathbb{Q}$ such that $qx + (1-q)a \in A$. Consider $p \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_p \in \mathbb{K}$, and $b_1, \ldots, b_p \in B$ such that $x = \lambda_1 b_1 + \cdots + \lambda_p b_p$, where B is a Hamel basis for X. Now let $\|\cdot\|$ be any norm on X. Again because A is linearly open we can find $l_1, \ldots, l_p \in \mathbb{L}$ such that

$$|l_i - \lambda_i| < \frac{\|x - a\|}{p \|b_i\|} \text{ for } 1 \leq i \leq p$$

and

$$q\left(l_1 b_1 + \cdots + l_p b_p\right) + (1-q)a \in A \cap Y.$$

We have the following:

$$\begin{aligned}
& \left\| x - \left(q\left(l_1 b_1 + \cdots + l_p b_p\right) + (1-q)a \right) \right\| \\
\leq\ & \left\| x - (qx + (1-q)a) \right\| \\
+\ & \left\| (qx + (1-q)a) - \left(q\left(l_1 b_1 + \cdots + l_p b_p\right) + (1-q)a \right) \right\| \\
=\ & (1-q)\|x - a\| + q|\lambda_1 - l_1|\|b_1\| + \cdots + q\left|\lambda_p - l_p\right|\left\|b_p\right\| \\
<\ & (1-q)\|x - a\| + q\|x - a\| \\
=\ & \|x - a\|.
\end{aligned}$$

As a consequence, $d_{\|\cdot\|}(x, A \cap Y)$ is never attained and thus A is anti-proximinal in X endowed with the norm $\|\cdot\|$. □

The reader may notice that the main ideas of the proof of Proposition 6.2 can be taken advantage of to show the following more general result.

Proposition 6.3.

1. *Let X be a metric space. If A is an anti-proximinal subset of X and B is a dense subset of A, then B is also anti-proximinal in X.*
2. *Let X be a vector space. If A is a totally anti-proximinal subset of X and B is a subset of A which is dense in A for any norm on X, then B is also totally anti-proximinal in X.*

The previous proposition serves to show counter-examples to many assertions on the properties of totally anti-proximinal sets when the condition of convexity is disregarded.

Example 6.4.

- *In [7, Theorem 3.4] it is proved that the union of any family of totally anti-proximinal sets is also anti-proximinal. This is not the case for the intersection of totally anti-proximinal sets. Indeed,*

$$(-1,1) \cap \mathbb{Q} \text{ and } (-1,1) \cap (\{0\} \cup \mathbb{R} \setminus \mathbb{Q})$$

  are both totally anti-proximinal subsets of $\mathbb{R}$, however their intersection is $\{0\}$ which is not totally anti-proximinal.

- *In [7, Theorem 3.6] several inheritance properties of totally anti-proximinal sets are studied. Here we will demonstrate that the total anti-proximinality is not hereditary to vector subspaces. Indeed, let $X := \mathbb{R}^2$ and consider $F := \mathbb{R} \times \{0\}$ and*

$$A := \left\{(x, y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) : x^2 + y^2 \leq 1\right\} \cup \{(0,0)\}.$$

  It is not difficult to check that A is a totally anti-proximinal non-convex subset of X. However, $A \cap F = \{(0,0)\}$ is not a generator system of F.

- *As we promised at the end of the previous section, we will finish this subsection by showing an example of a non-linearly open totally anti-proximinal set which has internal points. Let X be the real line and $A = (0,1) \cup (\mathbb{Q} \cap (0,2))$. Then A is a totally anti-proximinal subset of X such that* $\operatorname{inter}(A) \neq \varnothing$ *but A is not open.*

References

[1] R. Armario, F. J. García-Pacheco, and F. J. Pérez-Férnandez, On the Krein-Milman Property and the Bade Property, Linear Algebra and Applications 436 (2012) 1489–1502.
[2] N. Bourbaki, Topological vector spaces. Chapters 1–5. Translated from the French by H. G. Eggleston and S. Madan. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987.
[3] F. J. García-Pacheco, Four Non-Linear Problems on Normed Spaces. Volumen I. Verlag Dr. Müller, Berlin, 2008.
[4] F. J. García-Pacheco, Non-continuous linear functionals on topological vector spaces, Banach Journal of Mathematical Analysis 2 (2008) 11–15.
[5] F. J. García-Pacheco, An approach to a Ricceri's conjecture, Topology and its Applications 159 (2012) 3307–3313.
[6] F. J. García-Pacheco, Advances on the Banach-Mazur Conjecture for Rotations, Submitted.
[7] F. J. García-Pacheco and J. R. Hill, A partial positive solution to a Ricceri's conjecture, Submitted.
[8] B. Ricceri, Topological problems in nonlinear and functional analysis, Open problems in topology II, E. Pearl ed., 585–593. Elsevier, 2007.
[9] J. H. Webb, Countable-codimensional subspaces of locally convex spaces, Proceedings of the Edinburgh Mathematical Society 18 (1973) 167–171.